Appendix 1 The Skin Effect

At high frequencies the currents within a conductor tend to be restricted to the surface layer adjacent to the external electromagnetic fields. For simplicity we consider here the analysis for a plane slab of conducting material lying in the x–y plane and with current flowing in the x-direction, as illustrated in figure A1.1. Assume that the surface current density is $J\,\mathrm{A/m^2}$ and that the slab material has conductivity σ and permeability μ.

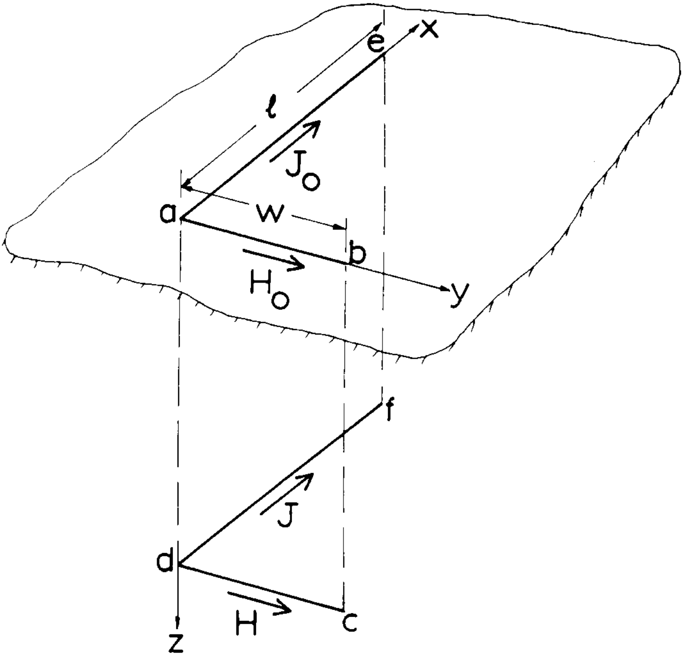

Figure A1.1 A plane solid conductor with current flowing in the x-direction

Now, Faraday's law can be written

$$\text{e.m.f.} = \oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{A1.1}$$

Consider the path of integration adfea. Conditions along sections ad and fe are identical, and so they will provide no net contribution to the integral. The electric field and current density are related by the expression

$$\boldsymbol{J} = \sigma \boldsymbol{E} \quad \text{or} \quad \frac{\boldsymbol{J}}{\sigma} = \boldsymbol{E} \tag{A1.2}$$

Therefore, equation A1.1 can be re-written

$$\frac{1}{\sigma}(J - J_0)l = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{A1.3}$$

But the total flux passing through the surface adfe is

$$\phi = \mu l \int_0^z H \,\mathrm{d}z \tag{A1.4}$$

Substituting in equation A1.3 and re-arranging yields

$$(J - J_0) = -\mu\sigma \frac{\mathrm{d}}{\mathrm{d}t}\int_0^z H \,\mathrm{d}z \tag{A1.5}$$

Assuming that the current and fields vary sinusoidally with time, and writing $J = J_m \mathrm{e}^{j\omega t}$, $H = H_m \mathrm{e}^{j\omega t}$, equation A1.5 becomes

$$(J_m - J_{0m}) = -\mathrm{j}\omega\mu\sigma \int_0^z H_m \,\mathrm{d}z \tag{A1.6}$$

and taking the derivative of both sides with respect to z

$$\frac{\mathrm{d}J_m}{\mathrm{d}z} = -\mathrm{j}\omega\mu\sigma H_m \tag{A1.7}$$

For the surface abcd Ampere's theorem gives

$$\oint \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{l} = I \tag{A1.8}$$

or

$$(H_0 - H)w = w\int_0^z J \,\mathrm{d}z \tag{A1.9}$$

and differentiating with respect to z we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}z} = -J \tag{A1.10}$$

or for sinusoidal variations of H and J

$$\frac{dH_m}{dz} = -J_m \tag{A1.11}$$

Differentiating equation A1.7 with respect to z and substituting from A1.11 to eliminate H_m yields

$$\frac{d^2 J_m}{dz^2} - j\omega\mu\sigma J_m = 0 \tag{A1.12}$$

This is a form of the wave equation (see section 3.1) for which the appropriate solution is

$$J_m = J_{0m} e^{-\sqrt{(j\omega\mu\sigma)}z} = J_{0m} e^{-(1+j)z\sqrt{(\omega\mu\sigma/2)}} \tag{A1.13}$$

or

$$J_m = J_{0m} e^{-jz/\delta} e^{-z/\delta} \tag{A1.14}$$

where δ is the *skin depth*, the distance over which the magnitude of the current falls to $1/e$ of its initial value, and is given by

$$\delta = \sqrt{\left(\frac{2}{\omega\mu\sigma}\right)} \tag{A1.15}$$

Equation A1.14 shows that the magnitude of the current density falls off exponentially with distance from the surface. An expression for the total current per unit width of surface can be obtained by integrating the current density with respect to depth.

Therefore

$$I = \int_0^\infty J\,dz \ \text{A/m} \tag{A1.16}$$

Substituting for J from equation A1.13 and integrating yields

$$I = \frac{J_0 \delta}{(1+j)} \tag{A1.17}$$

But at the surface

$$E_0 = \frac{J_0}{\sigma} \tag{A1.18}$$

and so the effective internal impedance of the surface for unit width and length is

$$Z_s = \frac{E_0}{I} = \frac{(1+j)}{\delta\sigma} = R_s + jX_s \tag{A1.19}$$

Therefore, the equivalent surface resistance for the conducting slab is

$$R_s = \frac{1}{\delta\sigma} = \sqrt{\left(\frac{\omega\mu}{2\sigma}\right)} \tag{A1.20}$$

which is the d.c. resistance for a slab of thickness σ. Thus, the a.c. resistance for a conducting slab with an exponential distribution of current is exactly the same as the d.c. resistance for a plane conductor of thickness equal to the skin depth δ.

For copper $\mu_r = 1$ and $\sigma = 58$ MS/m, so that the skin depth becomes

$$\delta = \frac{6.6 \times 10^{-2}}{\sqrt{f}} \text{m} \tag{A1.21}$$

and it is only 66 μm for a frequency of 1 MHz. Although the analysis outlined above applies for a plane conductor, the skin depth is so small for frequencies above the MHz region that the results apply for most practical conditions and conductor cross sections. (An exact solution is available for cylindrical conductors.[1]) The exponential function is such that the current falls to 1 per cent of its surface value in a distance of 4.6δ, and so from the electrical viewpoint the conductors need not be more than a few skin depths thick.